τ-Geometry: A Topological Derivation of α-Attractor Inflation with $\alpha = 2/3$

Adam Porybný*

Formalization and numerical verification: Claude (Anthropic)

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Abstract

We present a minimal axiomatic framework in which physical time is a complex variable $\tau \in \mathbb{H}^2$ (upper half-plane) and a photon is identified topologically with an elliptic curve of modular parameter τ. Under a meta-postulate of structural realism (observables as functions on isomorphism classes), modular invariance under $\mathrm{PSL}(2,\mathbb{Z})$ is not postulated but *derived* (Theorem 1), since the moduli space of elliptic curves over $\mathbb{C}$ is $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$. The hyperbolic field-space metric ($K = -1$) follows from $\mathrm{PSL}(2,\mathbb{R})$-invariance, fixing the α-attractor parameter to $\alpha = 2/3$ as a topological invariant of the modular curve $X(1)$. This yields a **parameter-free consistency relation**

$$r = 2(1 - n_s)^2,$$

giving $r \approx 0.0025$ for Planck 2018 $n_s = 0.9649$, testable by LiteBIRD (∼2032). Three additional formal results are presented: (i) Theorem 2 completeness (four force modes as topological necessity from $\mathrm{PSL}(2,\mathbb{Z})$ stabilizer classification); (ii) Theorem 3 (quaternion-frustration curvature, yielding $S^3 = \mathrm{SU}(2)$ with $\mathrm{Ric}_{ab} = 2\delta_{ab}$); (iii) P5 (a self-consistency theorem connecting π, via Lambert W, to $\mathrm{SL}(2,\mathbb{Z})$ via an explicit Möbius transformation).

1 Axioms

M0 (structural realism, meta-postulate) Physical observables are functions on isomorphism classes of states, not on their representatives.

A1 (complex time) The fundamental time variable is $\tau \in \mathbb{H}^2 = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

A2 (photonic space) A photon is topologically identified with $S^1 \times S^1$ equipped with complex structure τ — an elliptic curve over $\mathbb{C}$ with modular parameter τ.

A3 (observer projection) A classical observer measures $\mathrm{Re}(\tau)$ — the real projection of the complex substrate.

2 Theorem 1: Modular invariance

Theorem 1 (Modular invariance). *Under M0 + A1 + A2, all physical observables on the space of photon states are functions on the quotient* $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$.

Sketch. The moduli space of elliptic curves over $\mathbb{C}$ is $\mathcal{M}_{1,1} \cong \mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$ [2]. Two elliptic curves with τ-parameters related by a $\mathrm{PSL}(2,\mathbb{Z})$ transformation are isomorphic as complex tori. By A2 and M0, observables factor through this quotient. □

*Independent researcher, affiliated with the mezi4stěn social project, Brno, Czech Republic. Contact: adam.porybny@gmail.com

Consequence. The unique $\mathrm{PSL}(2,\mathbb{R})$-invariant Riemannian metric on $\mathbb{H}^2$ is the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

with constant Gaussian curvature $K = -1$. In the α-attractor framework of Kallosh and Linde [1], $K = -2/(3\alpha)$ implies $\alpha = 2/3$.

Topological verification via Gauss–Bonnet. The fundamental domain $\mathcal{F}$ of $X(1)$ has area

$$\mathrm{Area}(\mathcal{F}) = -2\pi\chi(\mathcal{X}_1) = \frac{\pi}{3},$$

where $\chi(\mathcal{X}_1) = -1/6$ is the orbifold Euler characteristic [3]. Hence

$$\alpha = \frac{2\,\mathrm{Area}(\mathcal{F})}{\pi} = \frac{2}{3} = -4\chi(\mathcal{X}_1),$$

a topological invariant, not a fit parameter.

3 Theorem 2: Force differentiation (completeness)

Theorem 2 (Completeness of force modes)**.** *The set of qualitatively distinct modulation modes of the substrate rotation on* $X(1) = \mathbb{H}^2/\,\mathrm{PSL}(2,\mathbb{Z})$ *is in bijection with*

$$\{\textit{orbit types of } \mathrm{PSL}(2,\mathbb{Z}) \textit{ on } \mathbb{H}^2\} \cup \{\textit{global orientation of } \mathrm{Im}(\tau)\}$$

and has exactly four elements.

Sketch. For elliptic $M \in \mathrm{SL}(2,\mathbb{Z})$ with $|\operatorname{tr}(M)| < 2$, the trace is integer, so $\operatorname{tr}(M) \in \{-1, 0, 1\}$. The characteristic polynomial $\lambda^2 - (\operatorname{tr})\lambda + 1 = 0$ gives eigenvalues $\pm i$ (trace 0, order 4 in SL, **2** in PSL) or $e^{\pm i\pi/3}$ (trace ± 1, order 6 in SL, **3** in PSL). By [4] Thm 2.5.6, finite stabilizers in Fuchsian groups are cyclic. Parabolic: one cusp at ∞, stabilizer generated by $T : \tau \to \tau + 1$. Orientation: $\mathbb{Z}_2$ choice of $\mathrm{Im}(\tau)$ sign. Total: 2 elliptic types + 1 parabolic + 1 orientation = 4. □

Attribution hypothesis (not yet formally proven, motivated by physical characteristics):

Mode	Feature	Force
Cusp ∞ (parabolic)	long-range, asymptotically free	Gravity
Elliptic i ($\mathbb{Z}_2$)	fermion spin-$\frac{1}{2}$ locus	Electromagnetism
Elliptic ρ ($\mathbb{Z}_3$)	three-fold color structure	Strong
Orientation ($\mathbb{Z}_2$)	only mode sensitive to P-violation	Weak

4 Theorem 3: Curvature from quaternion frustration

Theorem 3 (Curvature from quaternion frustration)**.** *Let X_a $(a = 1, 2, 3)$ be three vector fields on a smooth 3-manifold with Lie brackets*

$$[X_a, X_b] = 2\varepsilon_{abc}X_c.$$

Then the manifold admits a bi-invariant Riemannian metric with Ricci tensor $\mathrm{Ric}_{ab} = 2\delta_{ab}$ *and scalar curvature $R = 6$. It is diffeomorphic to $S^3 = \mathrm{SU}(2)$. Flat space is algebraically excluded.*

Proof. Structure constants: $f^c{}_{ab} = 2\varepsilon_{abc}$. The Killing form is

$$B_{ab} = f^c{}_{ad} f^d{}_{bc} = 4\varepsilon_{cad}\varepsilon_{dbc} = -8\delta_{ab}$$

(direct computation via ε-contraction identities). For a compact Lie group with bi-invariant metric, $\mathrm{Ric}(X,Y) = -\frac{1}{4}B(X,Y)$ [5, 6]. Hence $\mathrm{Ric}_{ab} = 2\delta_{ab}$, $R = 6$. Negative-definite B confirms $\mathrm{SU}(2) = S^3$. □

Physical interpretation. Field space is $\mathbb{H}^2$ (Theorem 1, $K = -1$); physical space supporting three anti-commuting quaternion directions must be S^3 ($K = +1$). The curvatures are complementary and sum to zero.

Cosmological consequence. With $\mathrm{Ric}_{ab} = 2g_{ab}$ and $R = 6$ in 3D, Einstein's equations in vacuum with cosmological constant yield $\Lambda_{\text{eff}} = 3H^2$. If the S^3 radius equals the Hubble radius today ($c/H_0 \approx 4.4\,\text{Gpc}$):

$$\frac{\Lambda_{\text{eff}}}{M_{\text{Pl}}^4} = \frac{3}{(M_{\text{Pl}} \cdot c/H_0)^2} \approx 4 \times 10^{-122},$$

an order-of-magnitude match with the observed $\Lambda_{\text{obs}}/M_{\text{Pl}}^4 \approx 10^{-122}$. The hierarchy problem reduces to: why is the S^3 radius of order Hubble? (A physical question, not fine-tuning.)

5 P5: CF band structure and self-consistency of π

For $x^x = c$ with $c \in (c_1, 4)$ where $c_1 = (3/2)^{3/2}$, the unique real solution in $(3/2, 2)$ is $x = e^{W(\ln c)}$, where W is the Lambert W function [7]. The continued fraction of x starts $[1; 1, a_2, \ldots]$ with

$$a_2 = \left\lfloor \frac{x-1}{2-x} \right\rfloor. \tag{1}$$

Möbius structure. Define $g(x) = (x-1)/(2-x)$. Its matrix is

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad \det M = 1 \quad \Rightarrow \quad M \in \mathrm{SL}(2, \mathbb{Z}).$$

Fixed point: $\varphi = (1+\sqrt{5})/2$ (golden ratio, solving $x^2 - x - 1 = 0$). Eigenvalues: $\varphi^{\pm 2}$. Trace: $3 = \mathrm{ord}(\rho)$. Band boundaries: $M^{-1}(n) = (2n+1)/(n+1)$.

Theorem 4 (P5 chain)**.** $a_2(\pi) = 5$.

Proof. Six-step chain across distinct branches of mathematics:

1. *Analysis:* $e^{i\pi} + 1 = 0$ defines π as the fundamental rotation constant.
2. *Topology:* $X(1)$ has elliptic stabilizers of orders $p = 2, q = 3$ (Theorem 2).
3. *Number theory:* CF mechanics gives $M \in \mathrm{SL}(2, \mathbb{Z})$ as above.
4. *Algebra:* $(p, q) = (2, 3)$ is the **unique** pair of natural numbers satisfying $(p-1)(q-1) = 2$, equivalent to $p + q + 1 = pq$. Hence $M^{-1}(5) = 11/6$, with denominator $p \cdot q = 6$.
5. *Arithmetic:* $(11/6)^{11/6} < \pi < (13/7)^{13/7}$, verified by Archimedean bounds $223/71 < \pi < 22/7$.
6. *Combination:* $a_2(\pi) = \lfloor g(x(\pi)) \rfloor = 5 = p + q = \mathrm{ord}(i) + \mathrm{ord}(\rho)$. □

Interpretation. The number π (periodicity of rotation) fed through self-reference (x^x) and the break-value Möbius transform $M \in \mathrm{SL}(2, \mathbb{Z})$ (an element of the same symmetry group whose orbifold $X(1)$ has elliptic orders 2 and 3) lands at band index equal to the sum of those orders. This is a *self-consistency theorem*: the rotation constant is compatible with its own symmetry group's topology.

6 Effective inflation and prediction

Kinetic sector (fully $\mathrm{PSL}(2,\mathbb{Z})$-invariant):

$$\mathcal{L}_{\mathrm{kin}} = -\frac{1}{2}\frac{(\partial\tau_1)^2 + (\partial\tau_2)^2}{\tau_2^2},$$

the Poincaré metric on $\mathbb{H}^2$, with $K = -1$, hence $\alpha = 2/3$.

Effective inflaton potential (T-model): along the geodesic $\tau_1 = 0$ with canonical field $\varphi = \ln \tau_2$:

$$V(\varphi) = \Lambda^4 \tanh^2\left(\frac{\varphi}{2}\right).$$

This is the standard $\alpha = 2/3$ T-model [1]. The microscopic derivation of $V(\varphi)$ from a fully modular-invariant fundamental action is listed as **Open problem P8** (§8).

Slow-roll analysis. End of inflation at $\epsilon_V = 2/\sinh^2(\varphi) = 1$: $\varphi_{\mathrm{end}} = \operatorname{arcsinh}(\sqrt{2}) \approx 1.146$. Numerical integration of $N_e = \int_{\varphi_{\mathrm{end}}}^{\varphi_*}(V/V_\varphi)\,d\varphi$ for $N_e = 57$ gives $\varphi_* \approx 5.44$, $\epsilon_V \approx 1.49 \times 10^{-4}$, $\eta_V \approx -0.017$, yielding $n_s \approx 0.965$ and $r \approx 0.00239$ — in agreement with the universal α-attractor formula

$$\boxed{r = 2(1 - n_s)^2 \approx 0.0025} \qquad (2)$$

to within 2% accuracy.

7 Falsifiability

Model	α	r ($n_s = 0.965$)
τ-geometry (this paper)	2/3	0.0025
Starobinsky R^2	1	0.0037
Goncharov–Linde	1/9	0.0004
KL attractor ($3\alpha = 7$)	7/3	0.0086

LiteBIRD sensitivity (∼2032): $\sigma(r) \approx 0.001$.

τ-geometry is falsified if: $r > 0.01$ or $r < 0.0005$ measured; or n_s inconsistent with Planck 2018; or any step of the derivation chain Theorem 1 $\to K = -1 \to \alpha = 2/3 \to r = 2(1-n_s)^2$ contains a formal error.

τ-geometry is validated (non-uniquely) if: $r \approx 0.0025$ is measured consistent with $\alpha = 2/3$ universality. Note that $\alpha = 2/3$ also arises in IIB axiodilaton sector of string theory; a positive detection would not uniquely select τ-geometry.

8 Open problems

- **P1 (partial).** Explicit FRW solution $a(t)$ beyond slow-roll approximation.
- **P2 (partial).** General GR limit from τ-geometry (Theorem 3 provides a concrete example).
- **P4.** Microscopic derivation of SM couplings from Theorem 2 modes (attribution hypothesis).
- **P6 (direction).** Formalization of preaxiom x^x as diagonal morphism in cartesian closed category, with Lawvere's theorem [8] forcing existence of elliptic fixed points $(2,3)$.
- **P7.** Microscopic determination of the S^3 radius governing Λ_{eff}.
- **P8.** Microscopic derivation of the effective T-model potential from a fully $\mathrm{PSL}(2,\mathbb{Z})$-invariant action.
- **P9.** Formal proof of the attribution hypothesis (cusp $\leftrightarrow$ gravity, etc.).

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The author welcomes technical criticism, collaboration, or 30 minutes of conversation.

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