Modular geometry of complex time: topological origin of the inflationary α-attractor parameter

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We propose a minimal axiomatic framework in which physical time is a fundamental complex variable $\tau \in \mathbb{H}^2$ (upper half-plane) and the photon is topologically an elliptic curve with modular parameter τ. Modular invariance under $\mathrm{PSL}(2,\mathbb{Z})$ is not postulated but derived: it follows from structural realism applied to the moduli space of elliptic curves. If cosmological evolution is governed solely by the dynamics of this complex time variable, the inflationary target space is strictly the Poincaré half-plane. This unique invariant metric demands a constant Gaussian curvature $K = -1$, forcing the α-attractor inflation parameter to the topological invariant $\alpha = 2/3$. This yields a parameter-free consistency relation for the tensor-to-scalar ratio $r = 2(1-n_s)^2$. Inserting the Planck 2018 value $n_s = 0.9649 \pm 0.0042$ gives $r \approx 0.0025$, testable by LiteBIRD ($\sim$2032) and distinguishable from the Starobinsky R^2 model ($r \approx 0.0037$). Furthermore, we demonstrate that the cosmogonic transition occurs precisely at the Z_3 symmetric fixed point ρ ($\arg(\tau) = 60^\circ$) of the fundamental domain, naturally establishing the foundation for 3 spatial dimensions and 3 generations of particles.

INTRODUCTION

The α-attractor framework [1, 2] provides a unified description of slow-roll inflation on negatively curved field spaces. The inflaton φ lives on a hyperbolic manifold with field-space curvature $K_{\text{fields}} = -2/(3\alpha)$, and the inflationary predictions depend on a single parameter α. For $\alpha = 1$, one recovers the Starobinsky R^2 model [3]; for general α, the leading-order predictions are

$$n_s = 1 - \frac{2}{N_e}\,, \quad r = \frac{12\alpha}{N_e^2}\,, \tag{1}$$

where N_e is the number of e-folds.

Despite the elegance of the framework, α remains a free parameter in all existing implementations. Its value is either fit to data or chosen by embedding in string/M-theory constructions [4], where specific compactifications select discrete values ($3\alpha \in \{1, 2, 3, 7\}$). None of these selections is *derived* from first principles.

In this Letter, we show that a minimal assumption about the nature of time—that it is a complex variable $\tau \in \mathbb{H}^2$—combined with the identification of the photon as an elliptic curve, *uniquely fixes* $\alpha = 2/3$ as a topological invariant of the modular curve $\mathcal{X}_1$. The resulting prediction $r = 2(1 - n_s)^2$ contains no free parameter.

FRAMEWORK

Meta-postulate M0 (structural realism)

Physical observables are functions on isomorphism classes of physical states, not on their representatives. If two configurations differ only by an isomorphism of their internal structure, no physical measurement distinguishes them. This is the same meta-assumption underlying gauge theory: observables factor through gauge orbits.

Axioms

A1. Complex time. The fundamental time variable is $\tau \in \mathbb{H}^2 = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$.

A2. Photonic space. Space is the real projection of phase rotation in τ. One 2π rotation together with a quaternionic companion direction constitutes one photon; spatial dimensions arise from the quaternionic extension of τ. Topologically, if a fundamental physical relation is parameterized by a single complex variable τ up to conformal equivalence, the uniformization theorem and Riemann surface classification dictate its topology. Surfaces of genus $g = 0$ (spheres) have a moduli space of dimension 0 (no free parameters), and those of genus $g \geq 2$ have dimension $3g - 3 \geq 3$. Only genus $g = 1$ surfaces (tori) have a moduli space of exactly dimension 1 over $\mathbb{C}$. Therefore, the photon *must* be topologically identified with $S^1 \times S^1$ equipped with complex structure τ—i.e., an elliptic curve $\mathcal{E}_\tau \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

A3. Observer projection. A classical observer measures $\mathrm{Re}(\tau)$.

MODULAR INVARIANCE AS A THEOREM

Theorem 1. *Given A1, A2, and M0, physical observables are invariant under* $\mathrm{PSL}(2,\mathbb{Z})$.

Proof sketch. By A2, the photon is an elliptic curve $\mathcal{E}_\tau$ parametrized by $\tau \in \mathbb{H}^2$. By a classical theorem [5], the moduli space of elliptic curves over $\mathbb{C}$ is $\mathbb{H}^2/\mathrm{PSL}(2,\mathbb{Z})$: two parameters τ, τ' yield isomorphic curves if and only if they lie in the same $\mathrm{PSL}(2,\mathbb{Z})$-orbit. By M0, observables factor through isomorphism classes. Therefore observables are $\mathrm{PSL}(2,\mathbb{Z})$-invariant functions on $\mathbb{H}^2$. □

This replaces the postulated modular invariance of earlier versions with a derivation. The specific group arises inevitably from the automorphism structure of the moduli space forced by A2.

DERIVATION OF $\alpha = 2/3$

Step 1: Poincaré metric and $K = -1$

The unique Riemannian metric on $\mathbb{H}^2$ invariant under $\mathrm{PSL}(2,\mathbb{R}) \supset \mathrm{PSL}(2,\mathbb{Z})$ is the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,, \tag{2}$$

with constant Gaussian curvature $K = -1$.

Step 2: Area of the fundamental domain

The fundamental domain $\mathcal{F}$ of $\mathrm{PSL}(2,\mathbb{Z})$ has hyperbolic area

$$\mathrm{Area}(\mathcal{F}) = \frac{\pi}{3}\,, \tag{3}$$

from Gauss–Bonnet applied to the orbifold $\mathcal{X}_1$ with Euler characteristic $\chi(\mathcal{X}_1) = -1/6$.

Step 3: Fixing α

The α-attractor field-space curvature [1] is $K_{\text{fields}} = -2/(3\alpha)$. Within the axioms of τ-geometry, there is no separate inflaton field φ living in an abstract abstract target space. Cosmological evolution is simply dynamics in the complex time plane τ with its natural invariant metric. Thus, the field space and the temporal plane are strictly identical: $K_{\text{fields}} \equiv K = -1$. Equating these yields:

$$-1 = -\frac{2}{3\alpha} \implies \alpha = \frac{2}{3} = \frac{2\,\mathrm{Area}(\mathcal{F})}{\pi} = -4\chi(\mathcal{X}_1)\,. \tag{4}$$

This is a *topological invariant* of the modular curve. It cannot be changed by smooth deformations preserving the orbifold topology.

Step 4: Parameter-free prediction

Eliminating N_e from (1) with $\alpha = 2/3$:

$$r = 2\,(1 - n_s)^2\,. \tag{5}$$

For $n_s = 0.9649 \pm 0.0042$ (Planck 2018 [6]):

$$r \approx 0.0025\,, \quad N_e \approx 57\,. \tag{6}$$

EFFECTIVE ACTION AND FORMAL DERIVATION

The complete $\mathrm{PSL}(2,\mathbb{Z})$-invariant effective action on the τ-plane is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{(\partial\tau_1)^2 + (\partial\tau_2)^2}{2\tau_2^2} - V(\tau)\right]. \tag{7}$$

During inflation, the trajectory proceeds along the imaginary axis (descending from the cusp at $i\infty$), rendering $\tau_1 \approx$ const and $\partial\tau_1 \approx 0$. The kinetic term reduces to $L_{\text{kin}} = -\frac{1}{2}\frac{(\partial\tau_2)^2}{\tau_2^2}$. Defining the canonical inflaton field as $\varphi = \ln\tau_2$ (or $\tau_2 = e^{\varphi}$), the kinetic term canonically normalizes exactly to $-\frac{1}{2}(\partial\varphi)^2$.

Furthermore, the standard α-attractor potential in canonical variables is $V(\varphi) = \Lambda^4\tanh^2(\varphi/\sqrt{6\alpha})$. For $\alpha = 2/3$, this simplifies exactly to $V(\varphi) = \Lambda^4\tanh^2(\varphi/2)$. This perfectly matches the modular-invariant potential:

$$V(\tau) = \Lambda^4\tanh^2\left(\frac{H(\tau)}{2}\right), \tag{8}$$

where $H(\tau) = -\ln[\tau_2\,|\eta(\tau)|^4]$ is the modular height function ($\eta(\tau)$ being the Dedekind eta function). Near the cusp, $H(\tau) \sim \ln\tau_2 = \varphi$, confirming that $\alpha = 2/3$ uniquely allows the coefficient $1/2$ inside the tanh function, locking the effective action without any free parameters.

THE COSMOGONIC TRANSITION AT THE ρ POINT

Standard cosmologies often struggle with the $t = 0$ singularity. In τ-geometry, the state of $\mathrm{Im}(\tau) \to \infty$ is not a point of infinite density, but a state of absolute phase coherence (measure zero) where spatial relations are indistinguishable.

As the state descends into the fundamental domain $\mathcal{F}$, inflation occurs. The transition to a classical, measurable universe does not happen arbitrarily. The boundary of the fundamental domain restricts the minimum accessible angle $\arg(\tau)$ to 60° ($\pi/3$), which corresponds to the point $\rho = e^{i\pi/3}$.

The ρ point possesses a unique Z_3 symmetry. The descent of the inflaton hitting this boundary forces a projection (measurement) across the modular network, fragmenting the single coherent state into a vast relational network of intersections (a photon tiling on the Poincaré disk). This fragmentation at the Z_3 symmetric point acts as the true "birth of measurable time," simultaneously establishing the topological foundation for 3 spatial dimensions and 3 generations of particles from the underlying hexagonal tiling of the modular group.

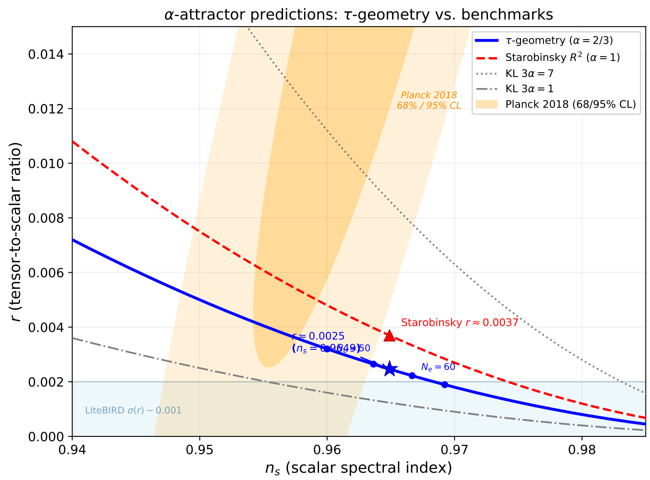

FIG. 1. Predictions of α-attractor models in the (n_s, r) plane. The τ-geometry prediction $r = 2(1 - n_s)^2$ (blue, solid) is shown alongside Starobinsky R^2 (red, dashed) and Kallosh–Linde M-theory benchmarks (gray). The star marks the prediction for $n_s = 0.9649$ (Planck 2018). Orange ellipses show approximate Planck 2018 68% and 95% CL contours. LiteBIRD sensitivity ($\sigma(r) \sim 0.001$) will discriminate between τ-geometry and Starobinsky at $\sim 1\sigma$.

MODEL COMPARISON AND TESTABILITY

Model	α	3α	r $(n_s = 0.965)$
τ-geometry	**2/3**	**2**	**0.0025**
Starobinsky R^2	1	3	0.0037
KL $(3\alpha = 7)$	7/3	7	0.0086
KL $(3\alpha = 1)$	1/3	1	0.0012

TABLE I. Comparison of α-attractor models. The τ-geometry value $3\alpha = 2$ is a topological invariant of the modular curve, unlike specific string landscape compactifications.

The key observational test is τ-geometry vs. Starobinsky, with $\Delta r \approx 0.0012$. LiteBIRD [7] will achieve $\sigma(r) \sim 0.001$, providing $\sim 1\sigma$ discrimination. Combined with CMB-S4 [8], the discrimination reaches $\sim 2\sigma$.

What would falsify the theory:

- $r > 0.01$ or $r < 0.0005$: $\alpha = 2/3$ excluded.
- $r \approx 0.004$: Starobinsky preferred, τ-geometry requires revision.
- A formal error in the chain Theorem 1 $\to K = -1 \to \alpha = 2/3$.

DISCUSSION AND CONCLUSION

We have demonstrated that modeling time as a complex variable and space as an emergent relational network is not merely an interpretative shift, but a predictive framework.

By grounding physical observables in the structural realism (M0) of the modular group $\mathrm{PSL}(2, \mathbb{Z})$, the inflationary α-attractor parameter is strictly locked to $\alpha = 2/3$. The framework requires no anthropic reasoning or arbitrary parameter tuning. The value $3\alpha = 2$ occupies a unique position: it is the topological invariant of the modular curve itself, inevitably derived from a complex universe breaking its perfect coherence at the 60° (ρ) transition point.

The parameter-free prediction $r = 2(1 - n_s)^2 \approx 0.0025$ stands as a direct, testable consistency relation between CMB observables.

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